Journal of Zankoy Sulaimani
Part-A- (Pure and Applied Sciences)
http://jzs.univsul.edu.iq

On the Boundedness of Solution of the Parabolic Differential Equation with Time Involution

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Article info

Original: 7 August 2020
Revised: 14 septembre 2020
Accepted: 26 September 2020
Published online: 20 December 2020

Key Words: Hilbert space, Boundedness, Involution, Parabolic equation.

Abstract

In the present paper, the initial value problem for the parabolic type involutory partial differential equation is investigated. Applying Green's function of space operator generated by the differential problem, we get formula for solution of this problem. The theorem on the existence and uniqueness of bounded solution of the nonlinear problem with involution is established.

1. Introduction

Differential equations with involution appear in mathematical models of ecology, biology, and population dynamics (see, e.g, [1]-[5] and the reference given therein). In recent decades, one-dimensional partial differential equations with involution have been investigated by many sciences in papers (see, e.g., [6]-[13] and the references given therein). In the study [7], the mixed problem of one dimensional parabolic equation with involution in x was investigated. Applying operator tools, the stability and coercive stability estimates in Hölder norms for the solution of this problem were established. In paper [6], a mixed problem for two dimensional elliptic equation with involution was studied. This problem was reduced to the boundary value problem for the abstract elliptic equation in the Hilbert space with a self-adjoint positive definite operator. Operator tools permit us to obtain stability and coercive stability estimates in Hölder norms, in one variable, for the solution. In paper [8], a stable difference scheme for approximate solution of an elliptic equations with involution was constructed. Theorem on stability and almost coercive stability and coercive stability of this difference scheme were established. The theoretical statements for statements for the solution of this difference scheme were supported by the results of the numerical experiment. In paper [10], a mixed problem of one dimensional hyperbolic equation with the involution in x was investigated. The stability estimates in maximum norm in t for the solution of this problem are established. In paper [13], the theory of the basis property of eigenfunctions of second order differential operators with involution was investigated, on this basis the Fourier method was justified for solving direct and inverse problems for one dimentional parabolic equations with involution in x. The applied value of these results in their importance in the study of several mathematical models containing partial differential equations with involution in space variable. The existence and uniqueness of the solution of a mixed problem for a parabolic equation with an involution in x in the form of a Fourier series were established. The classes of solvability of ill-posed problems for a parabolic equation with involution in x were considered. The questions of solvability of inverse problems for the heat equation and their fractional analogues were investigated. Solvability of inverse problems for a parabolic equation with an involution in x was proved.

As it mention before we need the values of unknown function at previous time for solving delay differential equations. There, there is impotant to study parabolic type differential equations with time involution. Noted that partial differential equations with time involution are not investigated before as well.

Our goal in this paper is to investigate the solution of the initial value problem for the parabolic type involutory partial differential equation

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} - b\frac{\partial^2 v(-t,x)}{\partial x^2} = f(t,x), \\ t, x \in I = (-\infty, \infty), \\ v(0,x) = \varphi(x), x \in I = (-\infty, \infty). \end{cases} \tag{1}$$

Here, $f(t,x)(t,x \in I)$ and $\varphi(x)(x \in I)$ are given smooth functions and $|b| > 1$. Applying Green's function of space operator generated by problem (1), we get formula for solution of this problem. In applications, the theorem on the existence and uniqueness of bounded solution of the nonlinear problem with involution is proved.

It is well-known that methods of finding Green's function of operator with involution and without involution are different. Therefore, we obtain formula of solution of problem (1) involving Green's function of operator with involution.

Problem (1) can be written as abstract initial value problem

$$\begin{cases} \frac{dv(t)}{dt} + Av(t) + bAv(-t) = f(t), t \in I, \\ \\ v(0) = \varphi \end{cases} \tag{2}$$

in a Hilber space $H = L_2(I)$ of all square integrable functions $\psi(x)$ defined on I equipped with the norm

$$\|\psi\|_{L_2(I)} = \left\{\int_{-\infty}^{\infty} \psi^2(x)dx\right\}^{1/2}.$$

Here, positive operator A defined by the formula

$$Au = -u''(x)$$

with domain $D(A) = \{u: u(x), u''(x) \in L_2(I)\}, f(t) = f(t,x)$ and $v(t) = v(t,x)$ are known and unknown abstract functions with values in $L_2(I)$ and $\varphi = \varphi(x)$ is the unknown element of $L_2(I)$. Now, we will obtain the initial value problem for the second order differential equation equivalent to problem (2) under smoothness conditions of solution.

Using initial condition and equation in problem (1), we get

$$v(0) = \varphi, v'(0) = f(0) - (1+b)A\varphi. \tag{3}$$

Differentiating equation (1), we get

$$v''(t) = bAv'(-t) - Av'(t) + f'(t). \tag{4}$$

Substituting $-t$ for t into equation (1), we get

$$v'(-t) = -bAv(t) - Av(-t) + f(-t). \tag{5}$$

Using these equations, we can eliminate $v(-t)$ and $v'(-t)$ terms. Actually, using equations (4) and (5), we get

$$v''(t) = -b^2A^2v(t) - bA^2v(-t) + bAf(-t) - Av'(t) + f'(t).$$

Using that and equation (1), we get

$$v''(t) = -b^2A^2v(t) + A[v'(t) + Av(t) - f(t)] + bAf(-t) - Av'(t) + f'(t)$$

or

$$v''(t) + (b^2-1)A^2v(t) = -Af(t) + bAf(-t) + f'(t). \qquad (6)$$

So, we get initial value problem (3) and (6) for the second order differential equation in a Hilbert $H = L_2(I)$. It is easy to see that

$$\frac{d^2v(t)}{dt^2} + (b^2-1)A^2v(t) = \left(\frac{d}{dt} + i\sqrt{b^2-1}A\right)\left(\frac{d}{dt} - i\sqrt{b^2-1}A\right)v(t).$$

Therefore, problem (3) and (6) can be written as abstract initial value problem

$$\begin{cases} \frac{d}{dt}v(t) + \sqrt{b^2-1}iAv(t) = u(t), v(0) = \varphi, \\ \frac{d}{dt}u(t) - \sqrt{b^2-1}iAu(t) = F(t), \\ F(t) = -Af(t) + bAf(-t) + f'(t), t \in I, \\ u(0) = f(0) - (1+b)A\varphi + i\sqrt{b^2-1}A\varphi \end{cases} \qquad (7)$$

for the system of first order abstract differential equations in a Hilbert $H = L_2(I)$. Integrating these equations, we can write

$$\begin{cases} v(t) = e^{-i\sqrt{b^2-1}tA}\varphi + \int_0^t e^{-i\sqrt{b^2-1}(t-y)A}u(y)dy, \\ u(t) = e^{i\sqrt{b^2-1}tA}\left[f(0) - (1+b)A\varphi + i\sqrt{b^2-1}A\varphi\right] \\ + \int_0^t e^{i\sqrt{b^2-1}(t-s)A}[-Af(s) + bAf(-s) + f'(s)]ds. \end{cases} \qquad (8)$$

Thus,

$$v(t) = e^{-i\sqrt{b^2-1}tA}\varphi + \int_0^t e^{-i\sqrt{b^2-1}(t-2y)A}dy\left[f(0) - (1+b)A\varphi + i\sqrt{b^2-1}A\varphi\right]$$

$$+ \int_0^t e^{-i\sqrt{b^2-1}(t-y)A}\int_0^y e^{i\sqrt{b^2-1}(y-s)A}[-Af(s) + bAf(-s) + f'(s)]dsdy.$$

Making the change of the order of integration and integration by parts, we can write

$$v(t) = \frac{1}{2}\left(e^{i\sqrt{b^2-1}tA} + e^{-i\sqrt{b^2-1}tA}\right)\varphi - \frac{1+b}{2\sqrt{b^2-1}}\left(e^{i\sqrt{b^2-1}tA} - e^{-i\sqrt{b^2-1}tA}\right)\varphi$$

$$+ \frac{1}{2i\sqrt{b^2-1}}\int_0^t \left(e^{i\sqrt{b^2-1}(t-s)A} - e^{-i\sqrt{b^2-1}(t-s)A}\right)[-f(s) + bf(-s)]ds \qquad (9)$$

$$-\frac{1}{2}\int_0^t \left(e^{i\sqrt{b^2-1}(t-s)A} + e^{-i\sqrt{b^2-1}(t-s)A}\right)f(s)ds.$$

2. Application

We consider the initial value problem for the nonlinear parabolic type involutory partial differential equation

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} - b\frac{\partial^2 v(-t,x)}{\partial x^2} = f(t,x,v(t,x)), \\ t, x \in I = (-\infty, \infty), \\ v(0,x) = \varphi(x), x \in I = (-\infty, \infty). \end{cases} \qquad (10)$$

Here, $f(t,x,v(t,x))(t,x \in I)$ and $\varphi(x)(x \in I)$ are given smooth functions and $|b| > 1$.We are intersted in studying the existence and uniqueness of bounded solution of problem (10) on $I \times I$. In general case of b the solution of (10) is not bounded on $I \times I$.

Theorem 2.1. Assume that $|b| > 1$ and f is continuous and bounded function on the region

$$P = \{(t,x,v): t,x \in I, \|v-\varphi\|_{L_2(I)} < M\}.$$

Suppose that f satisfies a Lipschitz condition on P with respect to its third argument, that is, there is a constant l such that for (t,x,u), $(t,y,v) \in P$

$$\|f(t,x,u) - f(t,x,v)\|_{L_2(I)} \le l\|u-v\|_{L_2(I)}. \quad (11)$$

Then, initial value problem (10) has a unique solution $v \in C(I \times I, L_2(I))$. This function v is the limit of the iterative sequence $\{v_n\}_{n=0}^{\infty}$ defined by the recursive Picard iteration formula

$$\begin{cases} \frac{\partial v_n(t,x)}{\partial t} - \frac{\partial^2 v_n(t,x)}{\partial x^2} - b\frac{\partial^2 v_n(-t,x)}{\partial x^2} = f(t,x,v_{n-1}(t,x)), \\ t,x \in I = (-\infty,\infty), \\ v_n(0,x) = \varphi(x), x \in I = (-\infty,\infty). n = 1,2,\dots, \end{cases} \quad (12)$$

where $v_0(t,x)$ is an arbitrary continuously differentiable function.

Proof. Problem (10) can be written as abstract initial value problem

$$\begin{cases} \frac{dv(t)}{dt} + Av(t) + bAv(-t) = f(t,v(t)), t \in I, \\ \\ v(0) = \varphi \end{cases} \quad (13)$$

in a Hilber space $H = L_2(I)$. Applying formula (9), we can write problem (13) in the equivalent integral form $v(t) = Tv(t)$. Here

$$Tv(t) = \frac{1}{2}\left(e^{i\sqrt{b^2-1}tA} + e^{-i\sqrt{b^2-1}tA}\right)\varphi - \frac{1+b}{2\sqrt{b^2-1}}\left(e^{i\sqrt{b^2-1}tA} - e^{-i\sqrt{b^2-1}tA}\right)\varphi$$

$$+\frac{1}{2i\sqrt{b^2-1}}\int_0^t \left(e^{i\sqrt{b^2-1}(t-s)A} - e^{-i\sqrt{b^2-1}(t-s)A}\right)[-f(s,v(s)) + bf(-s,v(s))]ds$$

$$-\frac{1}{2}\int_0^t \left(e^{i\sqrt{b^2-1}(t-s)A} + e^{-i\sqrt{b^2-1}(t-s)A}\right)f(s,v(s))ds.$$

Note that intergal form is a Volterra type integro-differential equation of the second kind. Therefore, applying the fixed-point theorem, we can complete the proof of Theorem 2.1.

Applying this approach we can obtain the existence and uniqueness of bounded solution of the initial boundary value problem

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} - \frac{\partial^2 v(t,x)}{\partial x^2} - b\frac{\partial^2 v\left(t-\frac{t_0}{2},x\right)}{\partial x^2} = f(t,x,v(t,x)), \\ t,x \in I = (-\infty,\infty), \\ v\left(\frac{t_0}{2},x\right) = \varphi(x), x \in I = (-\infty,\infty) \end{cases}$$

for the involutory nonlinear parabolic type partial differential equation on $I \times I$.

3. Conclusion

In the present paper, the initial value problem for the parabolic type involutory partial differential equation is investigated. Applying Green's function of space operator generated by this problem with time involution, we get formula for solution of this problem. In applications, the theorem on the existence and uniqueness of bounded solution of nonlinear problem with time involutionis was established. Moreover, applying the result of the monograph [14], the absolute stable difference schemes for the numerical solution of the initial value problem (10) for parabolic type involutory nonlinear partial differential equations can be constructed and investigated.

References

[1] C. E. Falbo. *"Idempotent differential equations*". Journal of Interdisciplinary Mathematics 6(3), 279–289 (2003).

[2] R. Nesbit. *"Delay Differential Equations for Structured Populations*". Structured Population Models in Marine,Terrestrial and Freshwater Systems. (1997).

[3] D. Przeworska-Rolewicz. *"Equations with Transformed Argument*". Algebraic Approach, Amsterdam, Warszawa. (1997).

[4] J. Wiener. *"Generalized Solutions of Functional Differential Equations*". Singapore New Jersey, London Hong Kong. (1993).

[5] A.Cabada and F. Tojo. *"Differential Equations with Involution's*". Atlantis Press. (2015).

[6] A. Ashyralyev and A. S. Erdogan.*"Well-Posedness of a Parabolic Equation with Involution".* Numerical Functional Analysis and Optimization. Vol. 38, No. 10, pp. 1295–1304. (2017).

[7] A. Ashyralyev and A. M. Sarsenbi. *"Well- Posedness of an Elliptic Equation with Involution".* Electronic Journal of Differential Equations, Vol. 284, No. 8, pp. 1072–6691. (2014).

[8] A. Ashyralyev, B. Karabaeva, and A. M. Sarsenbi. *"Green's function of the second order differential operator with involution".* AIP Conf. Proc 1759, 1063 (2016).

[9] A. Ashyralyev and A.Sarsenbi. *"Stability of a Hyperbolic Equation with the Involution".*Springer Proceedings in Mathematics and Statistics. Vol. 216, pp. 204–212. (2017).

[10] A.Sarsenbi. *"Well-Posedness of Mixed Problems for Parabolic type with Involution*". PhD Thesis, CSU, Kazakhstan. (2019).

[11] A. Ashyralyev and T. Abbas. *"A numerical algorithm for the involutory Schrödinger type problem".* AIP Conf. Proc. 2183, No. 3. (2019).

[12] A. Ashyralyev, A. Erdogan, and A. Sarsenbi. *"A note on the parabolic identification problem with involution and Dirichlet condition".* Bullition of the Karaganda Univesity-Mathematics Vol. 99, No. 3, pp. 130–139. (2020).

[13]M. Ashyraliyev, M. Ashyralyyeva, and A. Ashyralyev. *"A note on the hyperbolic-parabolic identification problem with involution and Dirichlet boundary condition".* Bullition of the Karaganda Univesity-Mathematics. Vol. 99, No. 3, pp. 120–129. (2020).

[14]A. Ashyralyev and P. E. Sobolevskii. *"New Difference Schemes for Partial Differential Equations*". Birkhäuser Verlag, Basel, Boston, Berlin. (2004).